The union of unit balls has quadratic complexity, even if they all contain the origin

Herv Brönnimann and Olivier Devillers*

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Abstract

We provide a lower bound construction showing that the union of unit balls in $\mathbb{R}^3$ has quadratic complexity, even if they all contain the origin. This settles a conjecture of Sharir.

1 Introduction

The union of a set of n balls in $\mathbb{R}^3$ has quadratic complexity $\Theta(n^2)$, even if they all have the same radius. All the already known constructions have balls scattered around, however, and Sharir posed the problem whether a quadratic complexity could be achieved if all the balls (of same radius) contained the origin.

In this note, we show a construction of n unit balls, all containing the origin, whose union has complexity $\Theta(n^2)$. As a trivial observation, we observe that the centers are arbitrarily close to the origin in our construction. In fact, if the centers are forced to be at least pairwise ε apart, for some constant $\varepsilon > 0$, then no more than $O(\frac{1}{\varepsilon^3})$ can meet in a single point, and hence the union has complexity at most $O(\frac{1}{\varepsilon^3}n) = O_\varepsilon(n)$. It is an interesting open question what a condition should be so that the union have subquadratic complexity and yet the balls have arbitrarily close centers.

By contrast, the *intersection* of n balls can have quadratic complexity if their radii are not constrained, but the complexity is linear if all the radii are the same [2]. Similarly, the convex hull of n balls can have also quadratic complexity [1], but that complexity is linear if they all have the same radius.

2 Construction

Let m and k be any integers. We define two families of unit balls: the first consists of k unit balls whose centers lie on a small vertical segment; the second consists of m unit balls whose centers lie on a small circle under the segment. (See Figure 3.) We show below that their union has quadratic $O(km)$ complexity.

The balls $B_1 \ldots B_k$. We denote by $B(p, r)$ the ball centered at p and of radius r. Let $n = k+m$ and P_i denote the point of coordinates $(0, 0, (i-1)/n^4)$, and $B_i = B(P_i, 1)$, for $i = 1, \ldots, k$. It is clear that the boundary of $\cup_{1\leq i\leq k} B_i$ consists of two hemispheres belonging to B_1 and B_k linked by a narrow cylinder of height less than $k/n^4 \leq 1/n^3$. This cylinder contains all the circles $\partial B_i \cap \partial B_{i+1}$ for $i = 1, \ldots, k-1$. (See Figure 1.)

*INRIA, BP 93, 06902 Sophia Antipolis, France. {Herve.Bronnimann|Olivier.Devillers}@sophia.inria.fr

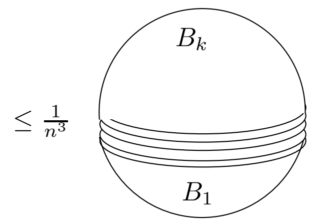

Figure 1: The union $\cup_{1\leq i\leq k} B_i$.

The balls $B_{k+1} \ldots B_{k+m}$. Let R be the point of coordinates $(x, 0, z)$ with

$$x = \frac{2n^2 - 4}{n^4}, \qquad z = -\frac{2n^2 - 4}{n^3}.$$

(Any values satisfying the constraints $P_k H < 1$ in (1) and $\ell < \frac{2}{n}$ in (2) below would do.) We define θ as the rotation around the z-axis of angle $2\pi/m$, and for each $j = 1, \ldots, m$, $R_{k+j} = \theta^{j-1}(R)$ and $B_{k+j} = B(R_{k+j}, 1)$.

3 Analysis

By our choice of x and z, we prove below that the boundaries of B_{k+1} and of the union $\cup_{i=1}^{k} B_i$ depicted in Figure 1 meet along a curve γ which satisfies the two claims below. The situation is depicted on Figure 2.

Claim 1 *The curve γ intersects all the balls B_i for $i = 0, \ldots, k$, .*

Claim 2 *The portion of γ which does not belong to B_1 (equivalently, which belongs to the union $\cup_{i=2}^{k} B_i$) is contained in an angular sector of angle at most $2\pi/m$.*

From claim 2, we conclude that the portion of γ which does not belong to B_1 is contained in the boundary of the union of the $n = k + m$ balls, From claim 1, we conclude that the portion of γ which does not belong to B_1 has complexity $\Omega(k)$. From claim 2, that it is contained in a small angular sector, hence appears completely on the boundary of the union of the $n = k+m$ balls, and it is replicated m times, once for each of the balls B_j, $j = 1, \ldots, m$. It follows that the union of all the balls B_i for $i = 1, \ldots, k + m$ has quadratic complexity $\Omega(km)$. Moreover, all the balls contain the origin. The union of the n balls is depicted on Figure 3.

The proofs involve only elementary geometry and trigonometry. The situation is depicted in Figure 4 and 5. Figure 4 depicts a section in the xz-plane of the spheres ∂B_i and ∂B_{k+1} and the point M, the highest point of intersection of the bounding spheres. The point M is also depicted on Figure 2.

Proof of Claim 1. It suffices to prove that M is higher than P_k, since then γ extends higher than P_k as well and passes through M by symmetry. The lowest point of γ belongs to B_1 and is clearly below the origin. The two facts together prove that γ must intersect all the balls between B_1 and B_k.

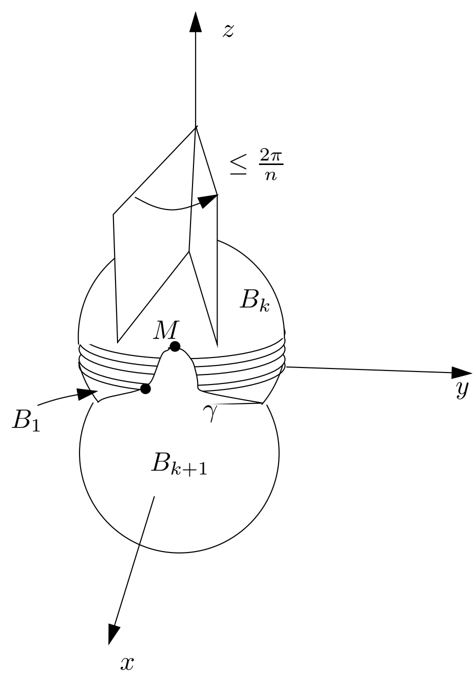

Figure 2: The union $\bigcup_{1\le i\le k} B_i \cup B_{k+1}$. The curve γ consists of a portion that belongs to B_1 and of another portion which is contained in a dihedral sector of angle less than π/m.

Let H be the point in the xz-plane on the median bisector of R and P_k, with same z-ordinate as P_k. (See Figure 4.) In order to prove that M is higher than P_k, it suffices to prove that H belongs to B_k, since then M is farther along the bisector. The two triangles QP_kH and KRP_k have equal angles, hence they are similar. It follows that

$$P_kH = P_kR\frac{P_kQ}{RK} = \frac{P_kR^2}{2RK} = \frac{x^2+(z-z_k)^2}{2x}, \tag{1}$$

where $z_k = \frac{k-1}{n^4}$. For x and z as given in the construction, we have

$$P_kH = 1/16\,\frac{-40\,n^4 - 15\,n^2 + 68 + 16\,n^6 - 16\,n^3 + 28\,n}{n^4\,(n^2-2)}$$

which is smaller than 1 for $n \geq 2$.

Proof of Claim 2. It is easy to see that the intersection of γ and a ball B_i $(2 \le i \le k)$ consists of at most two arcs of circle, any of which is monotone in angular coordinates around the z-axis, and that any such arc is entirely above the plane $z = 0$. Hence the intersections of γ with the xy-plane belong to B_1 and B_{k+1}. It suffices to show that these intersections are at a distance ℓ at most $\frac{2}{n} \le \sin\frac{\pi}{m}$ from the x-axis. (See Figure 5.)

In the xy-plane section, B_1 is a unit circle, and B_{k+1} is a circle of radius $r = \sqrt{1-z^2}$ and center R' of coordinates $(x, 0)$. (Recall that the center of B_{k+1} has coordinates $(x, 0, z)$.) Hence ℓ is the height of a triangle with base x and sides 1 and $r < 1$. It is elementary to compute that

$$\ell = \sqrt{1-\left(\frac{z^2+x^2}{2x}\right)^2}. \tag{2}$$

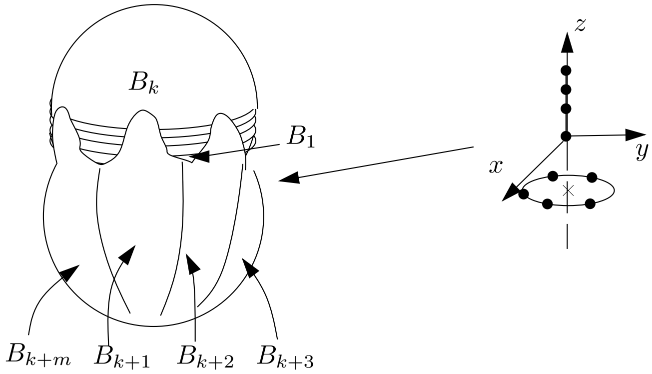

Figure 3: The union $\cup_{1\leq i\leq k+m} B_i$. On the right, a blow-up of the centers.

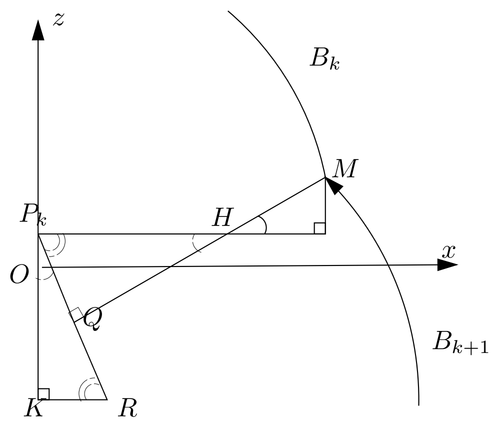

Figure 4: Figure for Claim 1.

For our choice or x and z, this yields

$$\ell = \sqrt{\frac{2\,n^6 + 3\,n^4 - 4\,n^2 - 4}{n^8}}$$

which is smaller than $2/n$ for $n \geq 2$.

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References

[1] Jean-Daniel Boissonnat, André Cérézo, Olivier Devillers, Jacqueline Duquesne, and Mariette Yvinec. An algorithm for constructing the convex hull of a set of spheres in dimension d. *Comput. Geom. Theory Appl.*, 6:123–130, 1996.

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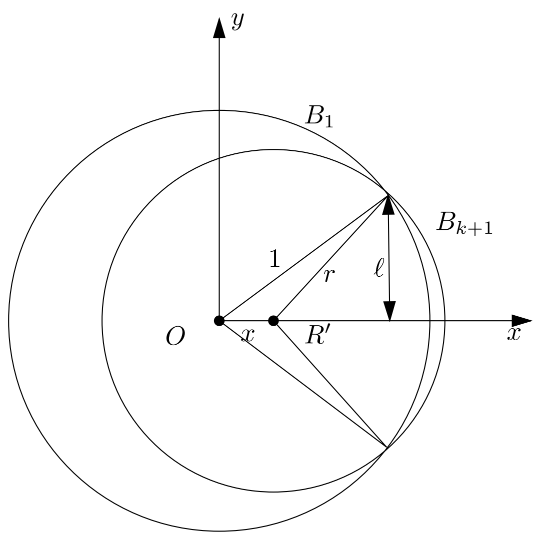

Figure 5: Figure for Claim 2.